Amplitude equations for isothermal double diffusive convection

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Amplitude equations are derived for isothermal double diffusive convection near threshold for both the stationary and oscillatory instabilities as well as in the vicinity of the codimension-2 point. The convecting fluid is contained in a thin Hele-Shaw cell that renders the system two dimensional, and convection is sustained by vertical concentration gradients of two species with different diffusion rates. The locations of the tricritical point for the stationary instability and the codimension-2 point are found. It is shown that these points can be made well separated (in the Rayleigh number R_s of the slow diffusing species) as the Lewis number varies. Hence the behavior near these points should be experimentally accessible. [S1063-651X(97)01705-4]

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Thermosolutal convection [1–5] and convection in binary mixtures heated from below [6–13] have been widely studied both theoretically [1–5,9–13] and experimentally [7,8]. They are examples of hydrodynamical systems whose behavior can be characterized by two diffusing scalars. In these systems, above a critical value of the thermal Rayleigh number, stationary or oscillatory convection arises depending on the value of the solutal Rayleigh number for the thermosolutal system or of the separation ratio for the binary mixture. Another system that exhibits these instabilities is isothermal double diffusive convection in a Hele-Shaw cell [14,15] with fixed concentration boundary conditions at the upper and lower boundaries of the experimental cell. This system is characterized by the competition between the destabilizing effect of a fast diffusing solute and the stabilizing effect of a slowly diffusing solute. These systems generically have a codimension-2 point (CTP) where a line of stationary instability intersects a line of oscillatory stability and a tricritical point (TRP) on the line of stationary instability near the CTP.

One of the differences [15] between the binary mixture and the isothermal double diffusion problem is the value of the Lewis number τ. For the binary mixture τ is the ratio of the diffusivity to the thermal diffusivity; for the isothermal problem it is the ratio of the diffusivity of the slower diffusing species to that of the faster. The value of τ for the two binary mixtures which have been extensively studied is of order 10^{-2} [12]. This leads to the theoretical prediction [12] that both the CTP and the TRP lie at separation ratios which are small in magnitude and negative. Experimentally, a change from a forward to a hysteretic-stationary bifurcation has been found at a positive value of separation ratio in the low temperature mixture [8]. In isothermal convection τ can be varied from 0.1 to nearly 1 by appropriately choosing the solutes [15]. We show below that these relatively large values of τ mean that the TRP and CTP occur at Rayleigh numbers (of the slowly diffusing species) of order 40 and that the relative separation ratio of these two points is of order one. Double diffusive convection in a Hele-Shaw cell should then be a promising system in which to experimentally investigate behavior near the TRP and CTP. We also present amplitude equations near threshold for both the stationary and oscillatory instabilities and around the point where these instabilities collide.

If $w/d \ll 1$ (w and d being the thickness and the height of a Hele-Shaw cell, respectively), then the system can be considered two dimensional, and in the Navier-Stokes equation the term $\nu\nabla^2\vec{u}$ can be replaced by $-12\nu\vec{u}/w^2$ as appropriate for this approximation [16] (ν is the kinematic viscosity). In the conducting state, the fluid is at rest, and the dimensional concentrations $\hat{c}_s$ and $\hat{c}_f$ read

$$\hat{c}_s = c_{s0}(1-\hat{z}/d), \qquad \hat{c}_f = c_{f0}(\hat{z}/d), \tag{1}$$

where d is the depth of the fluid and $\hat{z}$ the dimensional vertical coordinate. We nondimensionalize the hydrodynamic equations as in Ref. [15]. We assume that vertical velocity vanishes on the boundaries $z=0,1$ [6], and periodicity in the horizontal coordinate x. The typical experimental value of a modified Schmidt number $\sigma = 12\nu d^2/D_f w^2$ is of the order 10^5 [15]. Thus we work with the hydrodynamic equations in the limit of σ going to infinity. The basic dimensionless nonlinear equations describing the convective state in the Oberbeck-Boussinesq approximation with the Soret term [17] neglected can be written as [18]

$$\begin{pmatrix} \Delta & -R_s\partial_x & R_f\partial_x \\ \partial_x & -\tau\Delta+\partial_t & 0 \\ \partial_x & 0 & -\Delta+\partial_t \end{pmatrix} \vec{\xi} = \begin{pmatrix} 0 \\ J(\psi,C_s) \\ J(\psi,C_f) \end{pmatrix}, \tag{2}$$

where $\vec{\xi}(x,z)$ is a vector field with components ψ (stream function), and C_s and C_f are the deviation of the concentrations from the conduction profile. Δ and $J(f,g)$ are the Laplacian and Poisson bracket in the variables x and z, respectively. The Rayleigh numbers for the slow ($i=s$) and for the fast ($i=f$) species are defined by $R_i = \alpha_i c_{i0} g d w^2/12\nu D_f$, with g the acceleration due to gravity, ν the kinematic viscosity, and α_i the derivative of the logarithm of the density with respect to c_i.

Linear analysis of Eq. (2) yields the dispersion relation

$$\lambda^2+\left(k^2(1+\tau)+q^2\,\frac{R_s-R_f}{2k^2}\right)\lambda$$
$$+k^4\tau+q^2(R_s-\tau R_f)=0, \qquad (3)$$

where $k^2=\pi^2+q^2$. From Eq. (3) critical Rayleigh numbers for the stationary instability R_{fc}^{ss} and the oscillatory instability $R_{fc}^{\rm osc}$ are obtained. At their minima with respect to q, they are

$$R_{fc}^{ss}=\frac{R_s}{\tau}+4\pi^2, \qquad (4a)$$

$$R_{fc}^{\rm osc}=R_s+4\pi^2(1+\tau). \qquad (4b)$$

The critical wave number q is found to be π for both instabilities. For finite σ, there is a wave number difference between the stationary and oscillatory modes that is of order 10^{-4}. The frequency at onset is given by

$$\omega_o^2=-4\pi^4\tau^2+\pi^2(1-\tau)R_s, \qquad (5)$$

which vanishes at the CTP,

$$R_s^{c2}=\frac{4\pi^2\tau^2}{1-\tau}, \qquad R_f^{c2}=\frac{4\pi^2}{1-\tau}, \qquad (6)$$

as can be easily verified. The conduction state loses stability at a stationary bifurcation when R_f reaches R_{fc}^{ss}. The amplitude equation for the stationary branch has the form

$$\tau_0\partial_t A=\epsilon A-g_3A|A|^2. \qquad (7)$$

Here $\epsilon=(R_{fc}-R_{fc}^{ss})/R_{fc}^{ss}$. The nonlinear coefficient g_3 determines the bifurcation behavior of the system. To compute it, we expand the fields and the Rayleigh number in terms of a small parameter η,

$$R_f=R_{fc}^{ss}+\eta^2R_2^{ss}+\eta^4R_4^{ss}+\cdots, \qquad (8a)$$

$$\vec{\xi}=\frac{1}{2}\,(\eta\vec{\xi}_1A+\eta^2\vec{\xi}_2|A|^2+\eta^3\vec{\xi}_3A|A|^2+\cdots+\text{c.c.}), \qquad (8b)$$

and replace ∂_t by $\eta^2\partial_T$. Inserting these expansions into Eq. (2), we find a series of linear problems. At order η, the equation

$$\mathcal{L}_0\vec{\xi}_1\equiv\begin{pmatrix}\Delta & -R_s\partial_x & R_{fc}^{ss}\partial_x\\ \partial_x & -\tau\Delta & 0\\ \partial_x & 0 & -\Delta\end{pmatrix}\vec{\xi}_1=0 \qquad (9)$$

has the solution

$$\vec{\xi}_1=\begin{pmatrix} i\\ -1/2\pi\tau\\ -1/2\pi\end{pmatrix}\exp(-i\pi x)\sin(\pi z). \qquad (10)$$

At second order we find that the solution $|\vec{\xi}_2|\propto\sin(2\pi z)$. The integrability condition [17] involves the solution of the adjoint of Eq. (9), which is due to the non-Hermiticity of the operator $\mathcal{L}_0$, and at third order it yields Eq. (7), with g_3 and τ_0 given by

$$g_3=-\frac{1}{16R_{fc}^{ss}}\left(\frac{R_s}{\tau^3}-R_{fc}^{ss}\right),$$
$$\tau_0=-\frac{1}{2\pi^2R_{fc}^{ss}}\left(\frac{R_s}{\tau^2}-R_{fc}^{ss}\right). \qquad (11)$$

The TRP, the point at which g_3 vanishes, is found to be

$$R_s^{tc}=\frac{4\pi^2\tau^3}{1-\tau^2}. \qquad (12)$$

For $R_s<R_s^{tc}$, g_3 is positive and the bifurcation is supercritical and stable. Otherwise it is subcritical. At next order, a quintic term $-g_5A|A|^4$ is added to the right-hand side of Eq. (7). It is given by

$$g_5=\frac{1}{R_{fc}^{ss}}\left[\frac{g_3}{8}+\frac{3}{640}\left(\frac{R_s}{\tau^5}-R_{fc}^{ss}\right)-\frac{11g_3^2}{120\pi^2}\right]. \qquad (13)$$

At the TRP $g_5=3(1-\tau^2)/640\tau^2>0$, so that the bifurcation is forwards and $A\sim\epsilon^{1/4}$.

Using Eqs. (12) and (6) we compute the location of the TRP and the CTP when NaCl and propylene glycol are used as the fast and slow diffusing solutes, respectively. Their location is shown in the linear stability diagram in Fig. 1. The separation between these points in R_s is given by $R_s^{c2}-R_s^{tc}=4\pi^2\tau/(1-\tau)$.

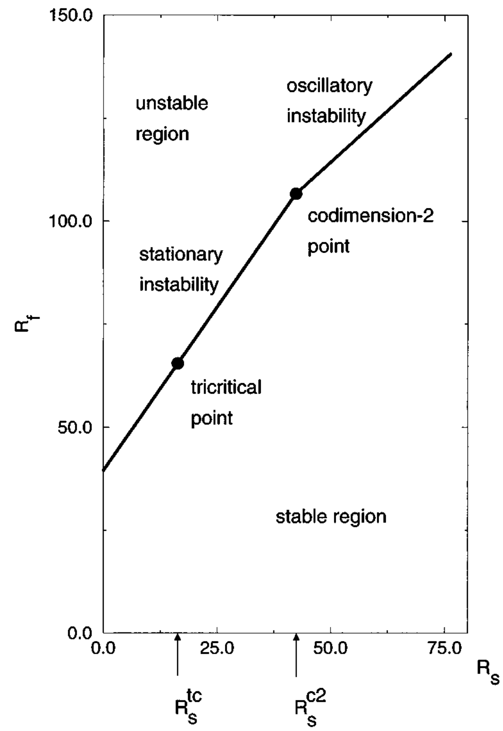

FIG. 1. Stability diagram for isothermal double diffusive convection as a function of the Rayleigh numbers of the fast and slow diffusing species, for $\tau=0.63$. This corresponds to NaCl and propylene glycol used as fast and slow diffusing solutes, respectively.

In order to derive amplitude equations for the oscillatory branch, we expand $\vec{\xi}$ and R_f as in Eq. (8) and time as $\partial_t = \partial_{t_0} + \eta^2 \partial_{t_1} + \cdots$. Inserting these expansions into the hydrodynamic equations (2), one finds a series of linear problems. If at first order in η, we select as a solution a superposition of right and left traveling waves (see Ref. [15]), at third order, the corresponding solvability conditions yield the complex amplitude equations

$$\tau_0^{\rm osc} \dot{A}_R = \epsilon^o (1 + i c_0) A_R - (K|A_R|^2 + M|A_L|^2) A_R , \tag{14a}$$

$$\tau_0^{\rm osc} \dot{A}_L = \epsilon^o (1 + i c_0) A_L - (K|A_L|^2 + M|A_R|^2) A_L \tag{14b}$$

[here $\epsilon^o \equiv (R_f - R_{fc}^{\rm osc})/R_{fc}^{\rm osc}$], whose coefficients are given by

$$4\tau_0^{\rm osc} = \frac{4}{R_{fc}^{\rm osc}}, \tag{15a}$$

$$c_0 = \frac{2\pi^2 \tau}{\omega_0}, \tag{15b}$$

$$K = \frac{i\pi^4}{2\omega_0 R_{fc}^{\rm osc}}, \tag{15c}$$

$$M = \frac{2\pi^6 (1+\tau)(\omega_0^2 + 4\pi^4 \tau)}{R_{fc}^{\rm osc}(\omega_0^2 + 4\pi^4)(\omega_0^2 + 4\pi^4 \tau^2)} + \frac{i\pi^4(\omega_0^4 - 16\pi^8 \tau^2)}{R_{fc}^{\rm osc}\omega_0(\omega_0^2 + 4\pi^4)(\omega_0^2 + 4\pi^4\tau^2)}. \tag{15d}$$

The real part of M is positive. The real part of K vanishes everywhere along the oscillatory branch, this is so even for finite σ as pointed out in Ref. [15]. This is also the case for thermohaline convection [21].

Near the CTP the two eigenvalues of the linear problem are near zero and the conductive state (1) becomes unstable against both the stationary and the oscillatory modes [20]. In the vicinity of this point the dynamics is described by [9,19]

$$\dot{A} = B,$$
$$\dot{B} = \mu_1 A + \mu_2 B + f(A,B), \tag{16}$$

where $f(A,B)$ is a nonlinear function. From the dispersion relation (3) it is readily seen that the unfolding parameters are given by

$$\mu_1 = \pi^2 \tau (R_f - R_{fc}^{ss}) \quad \text{and} \quad \mu_2 = \tfrac{1}{2}(R_f - R_{fc}^{\rm osc}). \tag{17}$$

We derived Eq. (16) following the method of Ref. [2]. In this method it is assumed that $\vec{\xi} = \vec{\xi}(x,z,A,\dot{A})$. Owing to periodicity in the x direction and the lack of distinction between left and right, Eq. (16) must be equivariant with respect to the group O(2), i.e., the group of reflections and rotations of a circle. This tells us how the form of $f(A,\dot{A})$ expanded in Taylor series should be

$$f = f_1 A|A|^2 + f_2 B|A|^2 + f_3 A^2 B^* + f_4 A|B|^2 + f_5 A^* B^2 + f_6 B|B|^2 + \cdots . \tag{18}$$

We also expand $\vec{\xi}$ in power series of A and B,

$$\vec{\xi} = \frac{1}{2}\left(\vec{\xi}_1 A + \vec{\phi}_1 B + \sum_{i,j,k,l} \vec{\xi}_{ijkl} A^i B^j A^{*k} B^{*l} + \text{c.c.} \right). \tag{19}$$

Inserting this expansion and Eq. (18) into Eq. (2) and equating the different powers of the amplitudes yields a sequence of problems that are to be solved order by order. At first order $\vec{\xi}_1$ is still given by Eq. (10) and $\vec{\phi}_1$ reads

$$\vec{\phi}_1 = \frac{1}{4\pi^2}\begin{pmatrix} 0 \\ 1/\tau^2 \\ 1 \end{pmatrix} \exp(-i\pi x)\sin(\pi z). \tag{20}$$

The coefficients of $f(A,B)$ appear in the third order equations, whose solvability conditions yield the values of the coefficients f_j ,

$$f_1 = \frac{\pi^4}{4}, \tag{21a}$$

$$f_2 = f_3 = -\frac{\pi^2}{8}\frac{1+\tau}{\tau}, \tag{21b}$$

$$f_4 = f_5 = -\frac{1+\tau^2}{16\tau^2}, \tag{21c}$$

$$f_6 = 0. \tag{21d}$$

The behavior of Eq. (16) in the vicinity of the CTP $\mu_1 = \mu_2 = 0$ can be classified in the (μ_1, μ_2) plane, and it is essentially the same as the one for binary mixture [6] or thermohaline convection [2]. We encounter a Hopf bifurcation at $\mu_2 = 0$ and a stationary bifurcation at $\mu_1 = 0$. Quadrants I $(\mu_1, \mu_2 > 0)$ and IV $(\mu_1 > 0,\ \mu_2 < 0)$ have one unstable fixed point which corresponds to the conducting state. In quadrant III $(\mu_1, \mu_2 < 0)$ the conducting state is stable and the fixed points $(B = 0, |A|^2 = -\mu_1/f_1)$ are unstable.

When $\mu_1 < 0$, the stability of $(B,A) = (0,0)$ depends on the value of μ_2. Starting in quadrant III, as μ_2 passes through zero, one finds a supercritical Hopf bifurcation and oscillatory convection is possible. As μ_2 is increased, the size of the limit cycle grows until it encounters the unstable fixed points at the value μ_{2c} where it disappears. This critical value where the oscillatory branch joins a branch of steady solutions is given by [2,6]

$$\mu_{2c} = -\frac{f_2 \mu_1}{5 f_1} = -\frac{1+\tau}{10\tau\pi^2}\, \mu_1 \equiv -\alpha(\tau)\mu_1 . \tag{22}$$

In the plane (R_s, R_f), this corresponds to a line L_c located above $R_{fc}^{\rm osc}$ but below R_{fc}^{ss} for $R_s > R_s^{c2}$. Specifically, the equation of L_c is given by

$$R_f = R_s\, \frac{1 + 2\alpha\pi^2}{1 + 2\alpha\pi^2\tau} + \frac{4\pi^2(1 + \tau + 2\alpha\pi^2\tau)}{1 + 2\alpha\pi^2\tau}. \tag{23}$$

Another way to find the coefficients of $f(A,B)$ [6] is by matching Eq. (16) with Eq. (14). Choosing $\eta^2 = \mu_2$ as a small parameter and inserting the expansion $\partial_t \to \partial_t + \eta^2 \partial_{t_1} + \cdots$ and the scaling of A,

$$A = \eta[A_R(t_1)\exp(i\omega_0 t) + A_L(t_1)\exp(-i\omega_0 t)], \quad (24)$$

into Eq. (16) leads at third order in η to two equations of the form (14), whose coefficients ($\widetilde{K}$ and $\widetilde{M}$) are given in terms of f_j. Comparing $\widetilde{K}$ and $\widetilde{M}$ with the coefficients K and M for $\omega_0 \ll 1$ (hence we expand M in Taylor series) one gets the relations

$$\frac{-if_1}{2\omega_0} + \frac{f_2 - f_3}{2} + (f_5 - f_4)\,\frac{i\omega_0}{2} = -K = -\frac{i\pi^4}{8\omega_0} \quad (25)$$

and

$$-\frac{if_1}{\omega_0} + f_3 - if_5\omega_0 = -M = -\frac{i\pi^4}{4\omega_0} - \frac{\pi^2(1+\tau)}{8\tau} + \frac{i\omega_0(1+\tau^2)}{16\tau^2} + O(\omega_0^2), \quad (26)$$

from which we can find the values of the coefficients f_j, and they agree with Eq. (21). Similarly it is possible to match f_1 with $g_3(R_s^{c2})$.

Equation (16) has been derived analytically in the presence of a $Z(2)$ reflectional symmetry for thermohaline [2] and binary mixture convection [6] in which f_4 and f_5 vanish. In the presence of O(2) symmetry, we have these coefficients that enable us to match Eq. (16) with the amplitude equations of the oscillatory branch.

In conclusion, the location of the TRP for the stationary branch and the CTP as a function of the Lewis number have been determined. We pointed out that since the TRP and the CTP are well separated, their location in the plane (R_s, R_f) should be determined in an experimental system. A quintic term for the stationary bifurcation was computed. Thus we are able to predict the crossover from the critical behavior in which $A \sim \epsilon^{1/2}$ to tricritical behavior in which $A \sim \epsilon^{1/4}$. We derived an amplitude equation that describes the dynamics of the system in the vicinity of the CTP. The most interesting event takes place when $\mu_1 < 0$ and μ_2 changes sign from negative to positive producing a limit cycle that disappears at μ_{2c}.

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